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LETTER TO THE EDITOR

Comment on the magnetic charge of Yang–Mills fields with large internal symmetry group

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Abstract. Working in the adjoint representation of the gauge group, we give a definition of magnetic charge in terms only of the Higgs field, for groups of $2n \times 2n$ matrices (n integer). We apply this recipe to two examples, SO(4) and SU(4).

The purpose of this letter is to give a generalization of the definition of magnetic charge made by t'Hooft (1974) for an SU(2) gauge theory with soliton solutions (t'Hooft 1974, Polyakov 1974) to theories with internal symmetries larger than SU(2).

In general we shall assume that finite-energy solutions to the gauge-field equations exist, but for the particular examples chosen, the existence of such solutions will be determined. Our aim is to give, in the general case, a definition of magnetic charge that depends only on the Higgs fields ϕ and not on the gauge-potentials A_μ.

We consider the Lagrangian density invariant under the compact gauge group $\mathscr{G}$

$$\mathscr{L} = \mathrm{Tr}[-\tfrac{1}{4}G_{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}(D_\mu\phi)(D_\mu\phi)] - V(\phi) \tag{1}$$

where the (ϕ, A_μ) fields are in the adjoint representation of $\mathscr{G}$ and therefore in the algebra of $\mathscr{G}$, $V(\phi)$ is an invariant potential function with the highest-power term of ϕ being quartic. The covariant derivative and the curvature are as usual defined by

$$D_\mu\phi = \partial_\mu\phi + \mathrm{i}e[A_\mu, \phi] \tag{2}$$

$$G_{\mu\nu} = \partial_{[\mu}A_{\nu]} + \mathrm{i}e[A_\mu, A_\nu]. \tag{3}$$

We shall be interested in solutions of the equations of motion arising from (1), which are subject to the *finite-energy condition* at infinity, and its consequences

$$D_\mu\phi = 0 \tag{4}$$

$$\partial V/\partial\phi = 0. \tag{5}$$

Following t'Hooft's definition of the electromagnetic field, and within the context of its generalization by Schwarz (1976), we generalize his expression to

$$F_{\mu\nu} = |\phi|^{-1}\,\mathrm{Tr}(\phi G_{\mu\nu} - (\mathrm{i}N/4e)|\phi|^{-2}\phi[D_\mu\phi, D_\nu\phi]) \tag{6}$$

where N is the dimension of the algebra of $\mathscr{G}$, and $|\phi|^2 = \mathrm{Tr}\,\phi^2$, which is taken to be a constant for large r, because of (4).

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As far as the magnetic charge μ is concerned, we are interested only in the value of the magnetic field $\boldsymbol{H}$ at large r, and

$$\mu = \frac{1}{4\pi}\int \boldsymbol{H}.\mathrm{d}\boldsymbol{S} = \frac{1}{8\pi}\int \epsilon_{ijk}F_{jk}\,\mathrm{d}S_i. \tag{7}$$

We wish to give an expression independent of A_μ for (7).

Using (4) then, it follows that for large r

$$F_{\mu\nu} = \partial_{[\mu}\,\mathrm{Tr}\,\hat{\phi}A_{\nu]} - \frac{\mathrm{i}N}{4e|\phi|^3}\mathrm{Tr}(\phi\,[\partial_\mu\phi,\partial_\nu\phi]) - \frac{ie}{|\phi|^3}\mathrm{Tr}\Big(|\phi|^2\phi A_{[\mu}A_{\nu]} + \frac{3N}{4}\phi A_{[\nu}\phi^2 A_{\mu]} + \frac{N}{4}\phi^3 A_{[\nu}A_{\mu]}\Big) \tag{8}$$

where $\hat{\phi} = |\phi|^{-1}\phi$ and we identify $B_\mu = \mathrm{Tr}\,\hat{\phi}A_\mu$ as the electromagnetic potential.

It is clear now that we shall have achieved our aim if the last term in equation (8) vanishes, in which case the electromagnetic field will indeed consist of a pure curl term and a 'magnetic current' piece that is independent of A_μ:

$$F_{\mu\nu} = \partial_{[\mu}B_{\nu]} - \frac{\mathrm{i}N}{4e}|\phi|^{-3}\,\mathrm{Tr}(\phi[\partial_\mu\phi,\partial_\nu\phi]). \tag{9}$$

One way of achieving this is to require the following conditions to be satisfied by the solutions of ϕ asymptotically:

$$\phi^{\mathrm{T}}\phi = \eta^2\mathbb{1} \tag{10a}$$

$$\phi^2 = \eta^2\mathbb{1} \tag{10b}$$

for $\mathscr{G} = \mathrm{SO}(N)$ and $\mathscr{G} = \mathrm{SU}(N)$ respectively. It is immediately obvious that these conditions could only be satisfied for *even* values of N, and even then only when they are compatible with the finite-energy conditions (5), where $V(0) > V_{\min} = 0$.

To illustrate the compatibility of (10) with (5), we consider the two special cases SO(4) and SU(4).

(*a*) $\mathscr{G} = \mathrm{SO}(4)$

A potential for which (5) is consistent with (10*a*) is

$$V(\phi) = \mathrm{Tr}(\phi^{\mathrm{T}}\phi - \eta^2\mathbb{1})^2 \tag{11a}$$

whose minimum $V_{\min} = 0$ is at $\phi^{\mathrm{T}}\phi = \mathbb{1}\eta^2$.

Here the solutions that satisfy (10*a*) asymptotically will actually have one unit of magnetic charge, for spherically symmetric solutions.

To see this, we look at the effect of condition (10*a*) on the two SU(2) fields ${}_{\pm}A_i$ that $\phi_{[\mu\nu]}$ ($\mu, \nu = 0, 1, 2, 3$) consists of:

$${}_{\pm}A_i = \tfrac{1}{2}(\phi_{[0i]} \pm \tfrac{1}{2}\epsilon_{ijk}\phi_{[jk]}), \qquad i = 1, 2, 3. \tag{12}$$

It follows that (10*a*) forces ***either*** ${}_{+}\boldsymbol{A}$ ***or*** ${}_{-}\boldsymbol{A}$ to vanish identically, hence reducing to the case of an SU(2) gauge field (t'Hooft 1974).

(*b*) SU(4)

A potential for which (5) and (10*b*) are consistent is

$$V(\phi) = \mathrm{Tr}(\phi^2 - \eta^2\mathbb{1})^2. \tag{11b}$$

In this case the orbit of the degenerate vacuum will be $\sqrt{2}\eta\lambda_{15}$, where λ_{15} is the SU(4) generator (in the adjoint representation)

$$\lambda_{15}=\frac{1}{\sqrt{2}}\begin{bmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{bmatrix}$$

which clearly satisfies (10*b*).

It then is only left to show that solutions of the gauge-field equations that satisfy (10*b*) asymptotically do exist. To this end we make the following spherically-symmetric *ansatz* for the solutions (ϕ, A_μ) in the manner of Tyupkin *et al* (1976):

$$A_0=0 \tag{13a}$$

$$\boldsymbol{A}=r^{-2}\alpha_1(r)\boldsymbol{x}\times\boldsymbol{\lambda}^{(1)}+r^{-2}\alpha_2(r)\boldsymbol{x}\times\boldsymbol{\lambda}^{(2)} \tag{13b}$$

$$\phi=r^{-1}\beta_1(r)\boldsymbol{x}\cdot\boldsymbol{\lambda}^{(1)}+r^{-1}\beta_2(r)\boldsymbol{x}\cdot\boldsymbol{\lambda}^{(2)}+\gamma(r)\lambda_{15}, \tag{13c}$$

where $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$ are the SU(4) generators corresponding to the two SU(2) subgroups of SU(4) respectively. The energy functional then is

$$\begin{aligned}\mathscr{E}(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma)&\\ =8\pi\int \mathrm{d}r[&(\alpha_1')^2+(\alpha_2')^2+2\alpha_1^2(1+e\alpha_1)^2+2\alpha_2(1+e\alpha_2)^2+\tfrac{1}{2}r^2(\beta_1')^2\\ &+\tfrac{1}{2}r^2(\beta_2')^2+\beta_1^2(1+2e\alpha_1)^2+\beta_2^2(1+2e\alpha_2)^2+\tfrac{1}{2}r^2(\gamma')^2+V(\phi)]\end{aligned} \tag{14}$$

where

$$V(\phi)=(\beta_1^2+\tfrac{1}{2}\gamma^2-\eta^2)^2+(\beta_2^2+\tfrac{1}{2}\gamma^2-\eta^2)^2+2\gamma^2(\beta_1^2+\beta_2^2).$$

That finite-energy solutions exist can be shown (Rawnsley 1977) by adapting the method of Tyupkin *et al* (1976) to this case, provided the following boundary conditions for large r are satisfied:

$$\begin{aligned}&\beta_1(r),\beta_2(r)\xrightarrow[r\to\infty]{}\pm\eta\\ &\gamma(r)\xrightarrow[r\to\infty]{}0.\end{aligned} \tag{15}$$

The magnetic charge corresponding to these solutions can be computed by use of (15), (13*c*), (9) and (7). The result is $e\mu=0,\pm1$. It is clear that taking β_1 (or β_2) and α_1 (or α_2) to be equal to zero in (13*b*, *c*), we should end up again with finite-energy solutions. The magnetic charges corresponding to these latter solutions are given by $e\mu=\pm\frac{1}{2}$.

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